Automated Derivation of Parametric Data Movement Lower Bounds for Affine Programs

Olivry et al

The approach

- They present some algorithms to compute I/O complexity lower bounds (same setup as HBL)
    - Their lower bounds are justified using arguments about standard computation graphs, which they call CDAGs
    - They generate them from a polyhedral model representation, though

Two types of lower bounds:

- HBL-style
- A way of counting a lot of live variables

Also there's something about "may-spill sets"

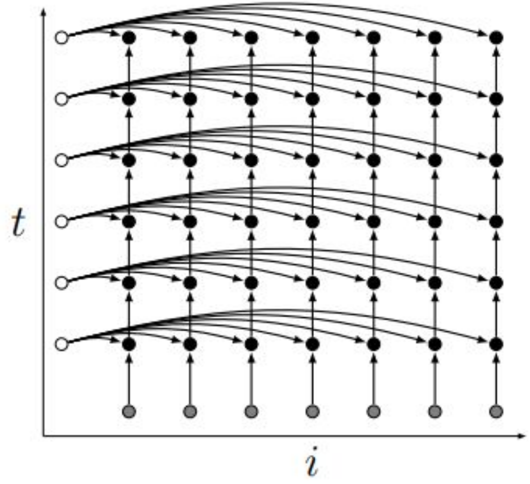
CDAGs

```
Parameters: N, M;
Input: A[N], C[M]; Output: A[N];
for(t=0;t<M;t++)
  for (i=0; i<N; i++)
    A[i] = A[i] * C[t];
```

(a) C-like code

Parameters: N, M;
Input: A[N], C[M]; Output: S_{M-1}[N];
for $(0 \leq t < M$ and $0 \leq i < N)$
  if (t==0): $S_{0,i}$ = A[i] * C[0];
  else: $S_{t,i}$ = $S_{t-1,i}$ * C[t];

(b) Corresponding single assignment form



(c) Corresponding CDAG. Input nodes A[N] (resp. C[N]) are shown in grey (resp. white) while compute node are shown in black

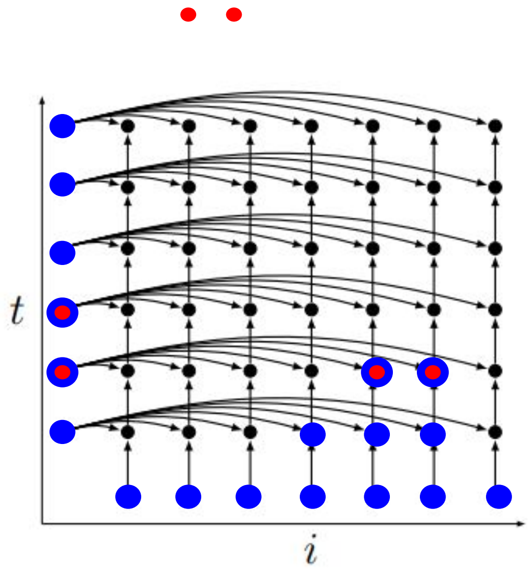
Fig. 1. Example 1

Pebble Games and S+T partitioning

Model computation with communication by playing a "pebble game" on the CDAG (place a red pebble to mean "in fast memory," blue to mean "in slow memory"; only have a fixed number of red pebbles to use)

Translate this into a sequence of computes and loads

If you have a fast memory of size S, and you prove that any k iterations require S+T inputs, you can partition the whole computation into N/k pieces and each must contain at least T loads



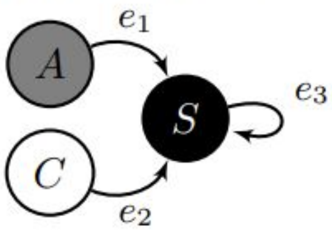
"Data flow graph"

```
for (0 ≤ t < M and 0 ≤ i < N)
  if (t==0): S[0,i]=A[i]*C[0];
  else: S[t,i]=S[t-1,i]*C[t];
```

(a) Single assignment form

$$D_A = [N] \to \{A[i]:\ 0 \le i < N\}$$
$$D_C = [N] \to \{C[t]:\ 0 \le t < M\}$$
$$D_S = [M, N] \to \{S[t, i]:\ 0 \le t < M \ \wedge\ 0 \le i < N\}$$
$$|D_S| = MN$$

(c) Node domains



(b) DFG

$$R_{e_1} = [N] \to \{A[i] \to S[0, i]:\ 1 \le i < N\}$$
$$R_{e_2} = [M, N] \to \{C[t] \to S[t, i]:\ 0 \le t < M \ \wedge\ 0 \le i < N\}$$
$$R_{e_3} = [M, N] \to \{S[t, i] \to S[t + 1, i]:\ 0 \le t < M - 1 \ \wedge\ 0 \le i < N\}$$

(d) Edge relations

Fig. 2. DFG for Example 1

Image taken directly from Olivry et al

Using the DFG

- Edges are direct dependencies, and are affine
- Paths/walks are composed dependencies

Find some paths/walks and argue that they correspond to projections between sets of iterations and sets of memory accesses; then apply HBL

- They identify two types of paths for which they can argue this

(2) Use the DFG representation to find a subset $V' \subseteq V$ and a set of projections (group homomorphisms) $\phi_1, \ldots, \phi_m$ with the property that:

(More details here)

$$\text{Any } K\text{-bounded set } P \subseteq V' \setminus \text{Sources}\,(V') \text{ satisfies } \left|\phi_j(\rho(P))\right| \leq K. \tag{4}$$

(3) Using ~~Theorem 3.10,~~ HBL derive an upper bound U on $|\rho(P)|$ for any K-bounded P. This provides a lower bound $\left\lceil \frac{|V' \setminus \text{Sources}(V)|}{U} \right\rceil$ on the number h of disjoint K-bounded sets in $V' \setminus \text{Sources}\,(V')$.

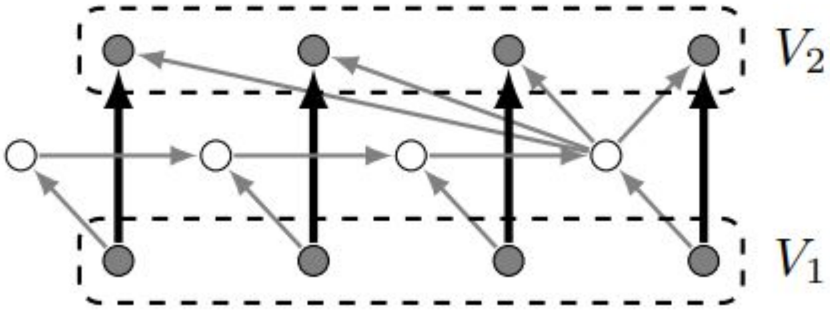
Using the DFG part 2

They also have a totally unrelated second lower bound they compute

If there are two disjoint sets of vertices in the CDAG V1 and V2, where everything in V1 has a path to V2, and I can find k disjoint paths between V1 and V2, then I need to do at least k-S loads from slow memory (if my fast memory has size S)

They find some special cases of this using the DFG

This is a bit strange and could maybe be extended usefully?



"May-spill sets"

Disclaimer: I don't fully understand this

General idea: If you have a fast memory of size S, you want to compute N iterations, and you know (through HBL, say) that only k iterations are computable given 2S values, you know you must have at least (N/k)*S memory accesses total

In effect, this argument "partitions" the CDAG into (N/k) disjoint sets of iterations, gets a lower bound (HBL-style, maybe) for each set, and sums them together

Can I also sum lower bounds together if the sets are not disjoint?

Olivry et al say: yes, if their "may-spill sets" are disjoint

"May-spill sets"

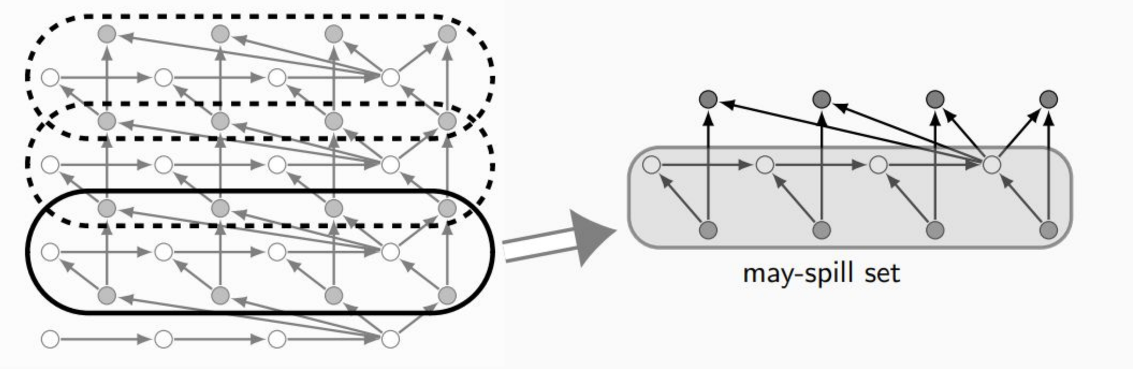
For a set V of iterations, its "may-spill set" is the set of iterations which have

1. both at least one incoming and at least one outgoing edge contained in V, or

2. at least 2 outgoing edges contained in V

Intuitively: The set of nodes in V which "may spill" their value to slow memory if we only have to compute V

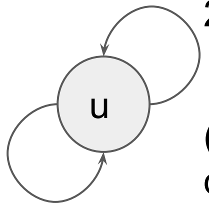
No communication with slow memory at all in "no-spill" set; no danger of double-counting



(c) Decomposition of the CDAG

Questions for ourselves

- Can we generalize this to get computable lower bounds on communication in parallel?
- Can we use a more general model (and thus get stronger bounds on a wider range of algorithms)?
- Since we know that expanding and low-diameter graphs have large lower bounds in parallel hardware, can we get characterizations of which polyhedral algorithms have these properties?



(2^k iterations at depth k)

(k+1 iterations at depth k)